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Free axisymmetric vibration of transversely isotropic piezoelectric circular plates

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Abstract

Based on three-dimensional elastic theory of piezoelectric materials, the axisymmetric state space formulation of piezoelectric laminated circular plates is derived. Finite Hankel transforms are used and the boundary variables in free terms are replaced, for two kinds of boundary conditions, to obtain ordinary differential equations with constant coefficients. Regarding the axisymmetric free vibration problem, two exact solutions for two different boundary conditions are found. Discarding piezoelectric effect, the exact solutions for transversely isotropic circular laminates are also obtained through the same procedure. Numerical examples are given and compared with those of Finite Element Method (FEM).

1. Introduction

The thin plate theory has been widely used in static and dynamic analyses of relatively thin plates, but it is not suitable for thick plates and laminates, particularly for higher modes of the plates. Mindlin (1951) published the plate theory which includes the effects of rotatory inertia and shear deformation. Based on this plate theory, Deresiewicz and Mindlin (1955) and Deresiewicz (1956) obtained the solutions of axisymmetric flexural vibration of a free circular disc and symmetric flexural vibration of a clamped circular disc, respectively. Kane and Mindlin (1956) presented the theory of high-frequency extensional vibration of circular plates. Reismann (1968) investigated the forced motion of a clamped circular plate using the Mindlin theory.

Srinivas and Rao (1970) exactly investigated the bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates based on a three-dimensional elasticity theory. Iyengar and Raman (1977) made an investigation on the free vibration of thick rectangular plates employing the method of initial function which was first introduced by Vlasov.

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Lee and Jiang (1996) employed the state-space-based method (i.e. the method of initial function) to exactly analyze the electroelastic behavior of piezoelectric laminated plates and obtained the exact solution of a simply supported rectangular plate with numerical examples presented. Chen et al. (1997) independently derived the exact solution of a thick piezoelectric rectangular plate also using the state-space-based method. Heyliger and Brooks (1995) presented the exact solutions for the free vibration behavior of simply-supported piezoelectric laminates in cylindrical bending for the case where the electrostatic potential or the normal electric displacement is specified to be zero at the upper and lower surfaces of the laminate. Heyliger and Saravanos (1995) developed the three-dimensional exact solutions for predicting the coupled electromechanical free vibration characteristics of simply supported laminated piezoelectric plates composed of orthorhombic layers. Batra et al. (1996) performed the analysis of a simply supported rectangular elastic plate forced into bending vibrations by the application of time harmonic voltages to piezoelectric actuators attached to its bottom and top surfaces by using the equations of linear elasticity. From the above mentioned papers, the exact solutions for simply supported rectangular or piezoelectric plates can be obtained by virtue of Fourier series which satisfied specified boundary conditions. For circular plates, some investigations on exact solutions for axisymmetric bending or vibrations have been published. Iyengar and Raman (1978) analyzed the free axisymmetric vibration of circular plates with arbitrary thickness also utilizing the method of initial function. Because the governing equation was an infinite-order differential equation, an investigation should be made on the approximate equation of the desired order after higher-order terms were ignored. Besides, the three-dimensional boundary conditions could not be exactly satisfied as in problems of simply supported rectangular plates. Celep (1978, 1980) also made three-dimensional investigation on the free axisymmetric vibration of circular plates using the method of initial function. In order to overcome the difficulty of algebraic manipulations of operators in cylindrical coordinates (r, θ, z), the following assumption was used

$$Gu = U = \frac{\mathrm{d}C(r)}{\mathrm{d}r}\bar{U}(z,\tau), \quad Gw = W = C(r)\bar{W}(z,\tau)$$

$$\sigma_z = Z = C(r)\bar{Z}(z,\tau), \quad \tau_{rz} = R = \frac{\mathrm{d}C(r)}{\mathrm{d}r}\bar{R}(z,\tau) \tag{1}$$

where u and w are displacement components in radial and axial directions, respectively, G is shear modulus, σ_z is axial normal stress, τ_{rz} is shear stress, $\bar{U}$, $\bar{W}$, $\bar{Z}$, $\bar{R}$ and $C(r)$ are unknown functions and $C(r)$ satisfies the following differential equation

$$\frac{\mathrm{d}^2C(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}C(r)}{\mathrm{d}r} + KC(r) = 0 \tag{2}$$

where $K = \pm k^2$ (k is an arbitrary constant). Fan and Ye (1990) adopted this assumption to investigate the axisymmetric vibration of transversely isotropic circular plates. Jianqiao Ye (1995) studied the axisymmetric buckling of homogeneous and laminated circular plates also using this assumption. However, it is found that assumption (1) imposes excessive restriction on the state variables, thus causing confusion in theory. For instance, according to eqns (1) and (2), the following three boundary conditions clamped: $U = W = 0, \quad \text{at } r = a$ (3a)

simply supported: $W = 0, \quad \sigma_r = 0, \quad \text{at } r = a$ (3b)

free: $\sigma_r = \tau_{rz} = 0, \quad \text{at } r = a$ (3c)

all result in $C = (\mathrm{d}C/\mathrm{d}r) = 0$ at $r = a$. Celep (1978, 1980) and Fan and Ye (1990) substituted the solution of eqn (2), $C = AJ_0(kr) + BI_0(kr)$, into the three-dimensional axisymmetric state space equations and derived a set of differential equations with constant coefficients. However, when $J_0(kr)$ and $I_0(kr)$ satisfied eqn (2), $K = k^2$ and $K = -k^2$ are obtained, respectively. Subsequently, substitution of $J_0(kr)$ and $I_0(kr)$ into governing equations gives distinct coefficient matrices. Hence the application of two-dimensional boundary conditions conjoining with the above mentioned form of C yields contradictions in deriving the characteristic equation.

From the above mentioned investigations, it is shown that there are inherent difficulties in applying the method of initial function in dynamic problems of anisotropic body. In fact, the three-dimensional exact solutions of free vibration of isotropic circular plates have not been found yet. This paper applies the finite Hankel transform to the axisymmetric dynamic equations of a piezoelectric circular plate and renders the free terms of the transformed question in terms of linear combination of boundary unknowns. Then the exact solutions for two boundary conditions, i.e. rigid slipping support and elastic simple support, are obtained, respectively. The numerical results are compared with those of the FEM and good agreement is displayed.

2. State space formulation and its solutions

Consider a p-ply piezoelectric circular laminate of radius a, thickness h. As shown in Fig. 1, the cylindrical coordinates (r, θ, z) are employed with the z-axis being along the symmetry axis of the circular laminate. Each layer has its own local coordinate z-axis as the jth lamina has (r, θ, z_j). h_j denotes the thickness of the jth lamina. The elastic symmetric axis of every lamina coincides with the z-axis.

The equations of motion for each of the layers are given by

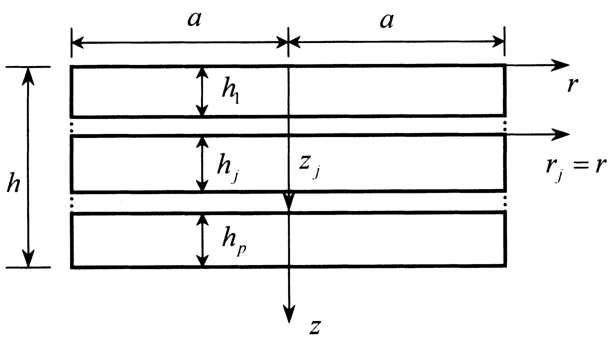


Fig. 1. Geometry and coordinate system of p-ply circular laminate.

$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=\rho\frac{\partial^2 u_r}{\partial \tau^2}$$

$$\frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=\rho\frac{\partial^2 w}{\partial \tau^2} \tag{4}$$

where σ_r, σ_θ, σ_z and τ_{rz} are four components of stress, w and u_r are axial and radial components of displacement, respectively, ρ denotes material density for a given layer and τ is time. The charge equation of electrostatics is given as

$$\frac{\partial D_r}{\partial r}+\frac{\partial D_z}{\partial z}+\frac{D_r}{r}=0 \tag{5}$$

where D_r and D_z are components of electric displacement. Each layer is assumed to be made of a transversely isotropic piezoelectric material with constitutive equations

$$\begin{aligned}
\sigma_r &= c_{11}s_r+c_{12}s_\theta+c_{13}s_z-e_{31}E_z\\
\sigma_\theta &= c_{12}s_r+c_{11}s_\theta+c_{13}s_z-e_{31}E_z\\
\sigma_z &= c_{13}s_r+c_{13}s_\theta+c_{33}s_z-e_{33}E_z\\
\tau_{rz} &= c_{44}\gamma_{rz}-e_{15}E_r\\
D_r &= e_{15}\gamma_{rz}+\varepsilon_{11}E_r\\
D_z &= e_{31}s_r+e_{31}s_\theta+e_{33}s_z+\varepsilon_{33}E_z
\end{aligned} \tag{6}$$

where s_r, s_θ, s_z and γ_{rz} are strain components, c_{11}, c_{12}, c_{13}, c_{33} and c_{44} are elastic stiffness components, e_{15}, e_{31} and e_{33} are the piezoelectric coefficients, ε_{11} and ε_{33} are the dielectric constants, E_r and E_z are electric-field components. The displacement components are related to the strain components and the electric-field components are related to the electrostatic potential ϕ through the relations

$$s_r=\frac{\partial u_r}{\partial r},\quad s_\theta=\frac{u_r}{r},\quad s_z=\frac{\partial w}{\partial z}$$

$$\gamma_{rz}=\frac{\partial w}{\partial r}+\frac{\partial u_r}{\partial z},\quad E_r=-\frac{\partial \phi}{\partial r},\quad E_z=-\frac{\partial \phi}{\partial z} \tag{7}$$

If the layer is not piezoelectric (i.e. $e_{15}=e_{31}=e_{33}=0$), electric and elastic fields are uncoupled and they can be solved separately. For the jth lamina, if u_r, σ_z, D_z, τ_{rz}, w and ϕ are chosen as state variables, the state space formulation can be written as

$$\frac{\partial}{\partial z_j}\begin{Bmatrix} u_r\\ \sigma_z\\ D_z\\ \tau_{rz}\\ w\\ \phi\end{Bmatrix}_j=\begin{bmatrix}0 & \mathbf{A}_j\\ \mathbf{B}_j & 0\end{bmatrix}\begin{Bmatrix} u_r\\ \sigma_z\\ D_z\\ \tau_{rz}\\ w\\ \phi\end{Bmatrix}_j \quad z_j\in[0,h_j] \tag{8}$$

where z_j is the local z-direction coordination of the jth lamina and

$$\mathbf{A}_j = \begin{bmatrix} \beta_1 & -\dfrac{\partial}{\partial r} & -\beta_2 \dfrac{\partial}{\partial r} \\ -\left(\dfrac{\partial}{\partial r}+\dfrac{1}{r}\right) & \rho \dfrac{\partial^2}{\partial \tau^2} & 0 \\ -\beta_2\left(\dfrac{\partial}{\partial r}+\dfrac{1}{r}\right) & 0 & \beta_6\left(\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}\right) \end{bmatrix} \tag{9}$$

$$\mathbf{B}_j = \begin{bmatrix} \rho \dfrac{\partial^2}{\partial \tau^2} - \beta_{12}\left(\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}-\dfrac{1}{r^2}\right) & -\beta_7 \dfrac{\partial}{\partial r} & -\beta_8 \dfrac{\partial}{\partial r} \\ -\beta_7\left(\dfrac{\partial}{\partial r}+\dfrac{1}{r}\right) & \beta_9 & \beta_{10} \\ -\beta_8\left(\dfrac{\partial}{\partial r}+\dfrac{1}{r}\right) & \beta_{10} & -\beta_{11} \end{bmatrix} \tag{10}$$

in which parameters β_i $(i = 1, 2, \ldots, 13)$ are defined as

$$\beta_1 = 1/c_{44}, \quad \beta_2 = e_{15}/c_{44}, \quad \beta_3 = 1/(c_{33}\varepsilon_{33}+e_{33}^2), \quad \beta_4 = c_{13}e_{33}-c_{33}e_{31}$$
$$\beta_5 = c_{13}\varepsilon_{33}+e_{31}e_{33}, \quad \beta_6 = \beta_2 e_{15}+\varepsilon_{11}, \quad \beta_7 = \beta_3\beta_5, \quad \beta_8 = \beta_3\beta_4, \quad \beta_9 = \beta_3\varepsilon_{33}$$
$$\beta_{10} = \beta_3 e_{33}, \quad \beta_{11} = \beta_3 c_{33}, \quad \beta_{12} = c_{11}-c_{13}\beta_7-e_{31}\beta_8, \quad \beta_{13} = \beta_{12}-c_{11}+c_{12}$$

The other derived variables are given by

$$D_r = \beta_2 \tau_{rz} - \beta_9 \frac{\partial \phi}{\partial r} \tag{11a}$$

$$\sigma_r = \beta_7 \sigma_z + \beta_8 D_z + \beta_{12} \frac{\partial u_r}{\partial r} + \beta_{13} \frac{u_r}{r} \tag{11b}$$

$$\sigma_\theta = \beta_7 \sigma_z + \beta_8 D_z + \beta_{13} \frac{\partial u_r}{\partial r} + \beta_{12} \frac{u_r}{r} \tag{11c}$$

When the circular plate vibrating with resonant frequency ω, the state variables can be assumed to be in the following form

$$u_r = u_r(r,z)\,\mathrm{e}^{i w \tau}, \quad \sigma_z = \sigma_z(r,z)\,\mathrm{e}^{i w \tau}, \quad D_z = D_z(r,z)\,\mathrm{e}^{i w \tau}$$
$$\tau_{rz} = \tau_{rz}(r,z)\,\mathrm{e}^{i w \tau}, \quad w = w(r,z)\,\mathrm{e}^{i w \tau}, \quad \phi = \phi(r,z)\,\mathrm{e}^{i w \tau} \tag{12}$$

Substituting the above expressions into eqn (8) and taking

$$\bar{u}_r = u_r/h, \quad \bar{\sigma}_z = \sigma_z/c_{11}^{(1)}, \quad \bar{D}_z = D_z/\sqrt{c_{11}^{(1)}\varepsilon_{33}^{(1)}}, \quad \bar{\tau}_{rz} = \tau_{rz}/c_{11}^{(1)}, \quad \bar{w} = w/h, \quad \bar{\phi} = \phi\frac{1}{h}\sqrt{\frac{\varepsilon_{33}^{(1)}}{c_{11}^{(1)}}}$$

$$\zeta = z_j/h, \quad \xi = r/a, \quad t = h/a, \quad d_j = h_j/h, \quad \Omega^2 = \rho^{(1)}\omega^2h^2/c_{11}^{(1)}, \quad \bar{\rho} = \rho/\rho^{(1)} \tag{13}$$

gives

$$\frac{\partial}{\partial\zeta}\begin{Bmatrix} \bar{u}_r(\xi,\zeta) \\ \bar{\sigma}_z(\xi,\zeta) \\ \bar{D}_z(\xi,\zeta) \\ \bar{\tau}_{rz}(\xi,\zeta) \\ \bar{w}(\xi,\zeta) \\ \bar{\phi}(\xi,\zeta) \end{Bmatrix}_j = \begin{bmatrix} 0 & \bar{\mathbf{A}}_j \\ \bar{\mathbf{B}}_j & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_r(\xi,\zeta) \\ \bar{\sigma}_z(\xi,\zeta) \\ \bar{D}_z(\xi,\zeta) \\ \bar{\tau}_{rz}(\xi,\zeta) \\ \bar{w}(\xi,\zeta) \\ \bar{\phi}(\xi,\zeta) \end{Bmatrix}_j \quad \zeta \in [0, d_j] \tag{14}$$

where

$$\bar{\mathbf{A}}_j = \begin{bmatrix} f_1 & -t\dfrac{\partial}{\partial\xi} & f_2\dfrac{\partial}{\partial\xi} \\ -t\left(\dfrac{\partial}{\partial\xi}+\dfrac{1}{\xi}\right) & -\bar{\rho}\Omega^2 & 0 \\ f_3\left(\dfrac{\partial}{\partial\xi}+\dfrac{1}{\xi}\right) & 0 & f_4\left(\dfrac{\partial^2}{\partial\xi^2}+\dfrac{1}{\xi}\dfrac{\partial}{\partial\xi}\right) \end{bmatrix} \tag{15}$$

$$\bar{\mathbf{B}}_j = \begin{bmatrix} -\bar{\rho}\Omega^2 + f_5\left(\dfrac{\partial^2}{\partial\xi^2}+\dfrac{1}{\xi}\dfrac{\partial}{\partial\xi}-\dfrac{1}{\xi^2}\right) & f_6\dfrac{\partial}{\partial\xi} & f_7\dfrac{\partial}{\partial\xi} \\ f_6\left(\dfrac{\partial}{\partial\xi}+\dfrac{1}{\xi}\right) & f_9 & f_{10} \\ f_{11}\left(\dfrac{\partial}{\partial\xi}+\dfrac{1}{\xi}\right) & f_{12} & f_8 \end{bmatrix} \tag{16}$$

and the non-dimensional parameters f_i ($i = 1, 2, \ldots, 13$) are defined as

$$f_1 = \beta_1 c_{11}^{(1)}, \quad f_2 = -\beta_2 t\sqrt{c_{11}^{(1)}/\varepsilon_{33}^{(1)}}, \quad f_3 = -\beta_2 t\sqrt{c_{11}^{(1)}/\varepsilon_{33}^{(1)}}, \quad f_4 = t^2\beta_6/\varepsilon_{33}^{(1)}$$

$$f_5 = -t^2\beta_{12}/c_{11}^{(1)}, \quad f_6 = -t\beta_7, \quad f_7 = -t\beta_8\sqrt{\varepsilon_{33}^{(1)}/c_{11}^{(1)}}, \quad f_8 = -\beta_{11}\varepsilon_{33}^{(1)}, \quad f_9 = \beta_9 c_{11}^{(1)}$$

$$f_{10} = \beta_{10}\sqrt{c_{11}^{(1)}\varepsilon_{33}^{(1)}}, \quad f_{11} = -t\beta_8\sqrt{\varepsilon_{33}^{(1)}/c_{11}^{(1)}}, \quad f_{12} = \beta_{10}\sqrt{c_{11}^{(1)}\varepsilon_{33}^{(1)}}, \quad f_{13} = t^2\frac{\beta_{13}}{c_{11}^{(1)}} \tag{17}$$

$c_{ij}^{(1)}$, $\rho^{(1)}$ and $\varepsilon_{ij}^{(1)}$ appearing in eqns (13) and (17) represent the elastic constants, material density and dielectric constants of the first lamina, respectively. The finite Hankel transform is defined as

$$\mathbf{J}_\mu[f(\xi,\zeta)] = \int_0^1 \xi f(\xi,\zeta) J_\mu(k\xi)\,\mathrm{d}\xi \tag{18}$$

where $J_\mu(k\xi)$ is the Bessel function of the first kind. Taking

$$\begin{aligned} &U(\zeta) = \mathbf{J}_1[\bar{u}_r(\xi,\zeta)], \quad \sigma(\zeta) = \mathbf{J}_0[\bar{\sigma}_z(\xi,\zeta)], \quad D(\zeta) = \mathbf{J}_0[\bar{D}_z(\xi,\zeta)] \\ &\tau(\zeta) = \mathbf{J}_1[\bar{\tau}_{rz}(\xi,\zeta)], \quad W(\zeta) = \mathbf{J}_0[\bar{w}(\xi,\zeta)], \quad \Phi(\zeta) = \mathbf{J}_0[\bar{\phi}(\xi,\zeta)] \end{aligned} \tag{19}$$

and applying the finite Hankel transform to eqn (14) gives

$$\frac{\mathrm{d}\mathbf{R}_j(\zeta)}{\mathrm{d}\zeta} = \mathbf{K}_j\mathbf{R}_j(\zeta) + \mathbf{Q}_j \quad \zeta \in [0, d_j] \tag{20}$$

where

$$\mathbf{R}_j(\zeta) = [U(\zeta) \quad \sigma(\zeta) \quad D(\zeta) \quad \tau(\zeta) \quad W(\zeta) \quad \Phi(\zeta)]_j^T \tag{21}$$

$$\mathbf{K}_j = \begin{bmatrix} 0 & \mathbf{K}_1 \\ \mathbf{K}_2 & 0 \end{bmatrix} \tag{22}$$

$$\mathbf{K}_1 = \begin{bmatrix} f_1 & kt & -f_2 k \\ -kt & -\bar{\rho}\Omega^2 & 0 \\ f_3 k & 0 & -f_4 k^2 \end{bmatrix} \tag{23}$$

$$\mathbf{K}_2 = \begin{bmatrix} -\bar{\rho}\Omega^2 - f_5 k^2 & -f_6 k & -f_7 k \\ f_6 k & f_9 & f_{10} \\ f_{11} k & f_{12} & f_8 \end{bmatrix} \tag{24}$$

and

$$\mathbf{Q}_j = \begin{Bmatrix} -t\bar{w}(1,\zeta)J_1(k) + f_2\bar{\phi}(1,\zeta)J_1(k) \\ -t\bar{\tau}_{rz}(1,\zeta)J_0(k) \\ f_3\bar{\tau}_{rz}(1,\zeta)J_0(k) - f_4\bar{E}_r(1,\zeta)J_0(k) + f_4 k\bar{\phi}(1,\zeta)J_1(k) \\ f_5[\bar{\varepsilon}_r(1,\zeta)J_1(k) - k\bar{u}_r(1,\zeta)J_0(k) + \bar{u}_r(1,\zeta)J_1(k)] + [f_6\bar{\sigma}_z(1,\zeta) + f_7\bar{D}_z(1,\zeta)]J_1(k) \\ f_6\bar{u}_r(1,\zeta)J_0(k) \\ f_{11}\bar{u}_r(1,\zeta)J_0(k) \end{Bmatrix} \tag{25}$$

in which

$$\bar{E}_r(\xi,\zeta) = -\frac{\partial\bar{\phi}(\xi,\zeta)}{\partial\xi}, \quad \bar{\varepsilon}_r(\xi,\zeta) = \frac{\partial\bar{u}_r(\xi,\zeta)}{\partial\xi} \tag{26}$$

Rendering eqns (11a,b) into non-dimensional form and taking $\xi = 1$ result in

$$f_4\bar{E}_r(1,\zeta) = t\bar{D}_r(1,\zeta) + f_3\bar{\tau}_{rz}(1,\zeta) \tag{27}$$

$$t\bar{\sigma}_r(1,\zeta) = -f_6\bar{\sigma}_z(1,\zeta) - f_7\bar{D}_z(1,\zeta) - f_5\bar{\varepsilon}_r(1,\zeta) + f_{13}\bar{u}_r(1,\zeta) \tag{28}$$

where

$$\bar{D}_r = D_r/\sqrt{c_{11}^{(1)}\varepsilon_{33}^{(1)}}, \quad \bar{\sigma}_r = \sigma_r/c_{11}^{(1)} \tag{29}$$

Substitution of eqns (27) and (28) into (25) gives

$$\mathbf{Q}_j = \begin{Bmatrix} -t\bar{w}(1,\zeta)J_1(k) + f_2\bar{\phi}(1,\zeta)J_1(k) \\ -t\bar{\tau}_{rz}(1,\zeta)J_0(k) \\ -t\bar{D}_r(1,\zeta)J_0(k) + f_4k\bar{\phi}(1,\zeta)J_1(k) \\ \left(\dfrac{c_{12}-c_{11}}{c_{11}^{(1)}}\right)t^2\bar{u}_r(1,\zeta)J_1(k) - f_5k\bar{u}_r(1,\zeta)J_0(k) - t\bar{\sigma}_r(1,\zeta)J_1(k) \\ f_6\bar{u}_r(1,\zeta)J_0(k) \\ f_{11}\bar{u}_r(1,\zeta)J_0(k) \end{Bmatrix} \tag{30}$$

It can be seen that the following two boundary conditions, named as elastic simple support and rigid slipping support conditions, respectively, will yield $\mathbf{Q}_j = \{0\}$.

(1) $\bar{w}(1,\zeta) = 0$, $\bar{\phi}(1,\zeta) = 0$, $[(c_{11}-c_{12})/c_{11}^{(1)}]t\bar{u}_r(1,\zeta) + \bar{\sigma}_r(1,\zeta) = 0$ and $J_0(k) = 0$
(2) $\bar{u}_r(1,\zeta) = 0$, $\bar{\tau}_{rz}(1,\zeta) = 0$, $\bar{D}_r(1,\zeta) = 0$ and $J_1(k) = 0$.

For these two conditions, eqn (20) becomes

$$\frac{d\mathbf{R}_j(\zeta)}{d\zeta} = \mathbf{K}_j\mathbf{R}_j(\zeta) \quad \zeta \in [0, d_j] \tag{31}$$

and its solutions can be written as

$$\mathbf{R}_j(\zeta) = \mathbf{T}_j(\zeta)\mathbf{R}_j(0) \quad \zeta \in [0, d_j] \tag{32}$$

where

$$\mathbf{T}_j(\zeta) = e^{\mathbf{K}_j\zeta} \tag{33}$$

By virtue of Cayley–Hamilton theorem which is available in the textbook by Deif (1982), $\mathbf{T}_j(\zeta)$ can be expressed as

$$\mathbf{T}_j(\zeta) = \alpha_0\mathbf{I} + \sum_{i=1}^{5}\alpha_i(\zeta)\mathbf{K}_j^i \tag{34}$$

where $\mathbf{I}$ is an identity matrix of order six, $\mathbf{K}_j$ is defined in eqn (22) and coefficients $\alpha_i(\zeta)$ $(i = 0, 1, \ldots, 5)$ can be determined from the following equation

$$\begin{Bmatrix} \alpha_0(\zeta) \\ \alpha_1(\zeta) \\ \alpha_2(\zeta) \\ \alpha_3(\zeta) \\ \alpha_4(\zeta) \\ \alpha_5(\zeta) \end{Bmatrix} = \begin{bmatrix} 1 & \eta_1 & \eta_1^2 & \eta_1^3 & \eta_1^4 & \eta_1^5 \\ 1 & \eta_2 & \eta_2^2 & \eta_2^3 & \eta_2^4 & \eta_2^5 \\ 1 & \eta_3 & \eta_3^2 & \eta_3^3 & \eta_3^4 & \eta_3^5 \\ 1 & \eta_4 & \eta_4^2 & \eta_4^3 & \eta_4^4 & \eta_4^5 \\ 1 & \eta_5 & \eta_5^2 & \eta_5^3 & \eta_5^4 & \eta_5^5 \\ 1 & \eta_6 & \eta_6^2 & \eta_6^3 & \eta_6^4 & \eta_6^5 \end{bmatrix}^{-1} \begin{Bmatrix} e^{\eta_1 \zeta} \\ e^{\eta_2 \zeta} \\ e^{\eta_3 \zeta} \\ e^{\eta_4 \zeta} \\ e^{\eta_5 \zeta} \\ e^{\eta_6 \zeta} \end{Bmatrix} \tag{35}$$

where η_i $(i = 1, 2, \ldots, 6)$ are the distinct eigenvalues of matrix $\mathbf{K}_j$. When equal eigenvalues occur, eqn (34) will take on other forms.

In eqn (32), taking $\zeta = d_j$ yields

$$\mathbf{R}_j(d_j) = \mathbf{T}_j(d_j)\mathbf{R}_j(0) \tag{36}$$

This formulation establishes the relationship between the jth lamina's physical quantities of the upper and lower surfaces by the transfer matrix $\mathbf{T}_j(d_j)$. In eqn (36), taking $j = 1,\ 2, \ldots, p$ and considering the continuity conditions of u_r, σ_z, D_z, τ_{rz}, w and ϕ at interfaces as follows:

$$\mathbf{R}_{j+1}(0) = \mathbf{R}_j(d_j) \tag{37}$$

one has

$$\mathbf{R}_p(\zeta_p) = \mathbf{F}\mathbf{R}_1(0) \tag{38}$$

where

$$\mathbf{F} = [F_{kl}] = \prod_{j=p}^{1} \mathbf{T}_j(d_j) \tag{39}$$

For the free vibration problem, the boundary conditions at the bottom and top surfaces of the laminate can be written as:

$$\sigma(d_p) = \tau(d_p) = \sigma(0) = \tau(0) = 0, \quad D_z(d_p) = D_z(0) = 0 \quad \text{for Case 1} \tag{40}$$

and

$$\sigma(d_p) = \tau(d_p) = \sigma(0) = \tau(0) = 0, \quad \phi(d_p) = \phi(0) = 0 \quad \text{for Case 2} \tag{41}$$

or

$$\sigma(d_p) = \tau(d_p) = \sigma(0) = \tau(0) = 0, \quad D_z(d_p) = 0, \quad \phi(0) = 0 \quad \text{for Case 3} \tag{42}$$

Substituting eqns (40), (41), or (42) into eqn (38), respectively, yields

$$\begin{bmatrix} F_{21} & F_{25} & F_{26} \\ F_{31} & F_{35} & F_{36} \\ F_{41} & F_{45} & F_{46} \end{bmatrix} \begin{Bmatrix} U(0) \\ W(0) \\ \phi(0) \end{Bmatrix} = 0 \tag{43}$$

or

$$\begin{bmatrix} F_{21} & F_{23} & F_{25} \\ F_{41} & F_{43} & F_{45} \\ F_{61} & F_{63} & F_{65} \end{bmatrix} \begin{Bmatrix} U(0) \\ D(0) \\ W(0) \end{Bmatrix} = 0 \tag{44}$$

or

$$\begin{bmatrix} F_{21} & F_{23} & F_{25} \\ F_{31} & F_{33} & F_{35} \\ F_{41} & F_{43} & F_{45} \end{bmatrix} \begin{Bmatrix} U(0) \\ D(0) \\ W(0) \end{Bmatrix} = 0 \tag{45}$$

Setting the determinant of coefficients matrices of homogeneous eqns (43)–(45) to zero for non-trivial solutions yields the characteristic frequency equations. The frequency equations are transcendental and give an infinite number of frequencies for each k, which cannot be obtained by classical theories of plate.

3. Calculating frequency and mode shape

For the elastic simple support condition, the serial roots k_i $(i = 1, 2, \ldots)$ of equation $J_0(k) = 0$ must be found and substitution of k_1, $k_2, \ldots$ one by one into eqns (22)–(24), as well as elastic constants, piezoelectric coefficients, dielectric constants, material densities and geometric parameters of the laminate, yields the expression of $\mathbf{K}_j$ $(j = 1, 2, \ldots, p)$. After $\mathbf{T}_j(d_j)$ is evaluated from eqns (34) and (35), the matrix elements F_{ij} of eqns (43)–(45) are obtained from eqn (39) and the non-dimensional frequency Ω becomes the only unknown in the frequency equations. To find the solutions of frequency equations, the frequency Ω is stepped through a sequence of small increments and the sign of the determinant is computed for each value and recorded. After a sufficient number of sign crossings have been identified, the values for Ω that yields a zero determinant can be isolated and refined using bisection. The sign change is monitored by computing the values of the determinants using the Gauss elimination method. Once the non-dimensional frequency Ω is obtained, substituting it into eqns (43), (44) or (45) yields the ratios of $U(0)$ and $W(0)$ to $D(0)$ or $\phi(0)$, respectively. By virtue of the inverse Hankel transform formulations given by Sneddon (1951), the corresponding mode shape is obtained as

$$\bar{u}_r(\xi, \zeta) = 2\sum_i U(k_i, \zeta) \frac{J_1(k_i \xi)}{[J_1(k_i)]^2}$$
$$\bar{w}(\xi, \zeta) = 2\sum_i W(k_i, \zeta) \frac{J_0(k_i \xi)}{[J_1(k_i)]^2} \tag{46}$$

For rigid slipping support conditions, variables k_i $(i = 1, 2, \ldots)$ are roots of equation $J_1(k) = 0$. The following procedure is similar to those of foregoing elastic simple support conditions and the mode shape is determined by

$$\bar{u}_r(\xi, \zeta) = 2\sum_i U(k_i, \zeta) \frac{J_1(k_i \xi)}{[J_0(k_i)]^2}$$

$$\bar{w}(\xi,\zeta) = 2\sum_i W(k_i,\zeta)\frac{J_0(k_i\xi)}{[J_0(k_i)]^2} \tag{47}$$

In fact, the corresponding k_i of the specified Ω has only one and so the right side of eqns (46) and (47) have only one term.

4. Transversely isotropic circular plate

If the layer is not piezoelectric, the piezoelectric coefficients e_{15}, e_{31} and e_{33} vanish. Then in eqns (9) and (10), the parameters β_2, β_4, β_8 and β_{10} become zero with β_3, β_5, β_6 and β_{12} being simplified. Subsequently, eqns (9) and (10) can be rendered in terms of block-diagonal matrices as well as eqns (23) and (24) because of $f_2 = f_3 = f_7 = f_{10} = f_{11} = f_{12} = 0$ from eqn (17). Meanwhile, eqns (11), (25), (27) and (28) are also simplified. For the two boundary conditions defined as above, i.e. elastic simple support and rigid slipping support conditions, eqn (31) can be uncoupled as

$$\frac{d\mathbf{R}_j^*}{d\zeta} = \mathbf{K}_j^*\mathbf{R}_j^*(\zeta) \tag{48}$$

$$\frac{d\mathbf{R}^{**}_j}{d\zeta} = \mathbf{K}^{**}_j\mathbf{R}^{**}_j(\zeta) \tag{49}$$

where

$$\mathbf{R}_j^* = [U(\zeta) \quad \sigma(\zeta) \quad \tau(\zeta) \quad W(\zeta)]_j^T, \quad \mathbf{R}^{**}_j = [D(\zeta) \quad \phi(\zeta)]^T \tag{50}$$

$$\mathbf{K}_j^* = \begin{bmatrix} 0 & \mathbf{K}_{1j}^* \\ \mathbf{K}_{2j}^* & 0 \end{bmatrix}, \quad \mathbf{K}^{**}_j = \begin{bmatrix} 0 & -f_4k \\ f_8 & 0 \end{bmatrix} \tag{51}$$

$$\mathbf{K}_{1j}^* = \begin{bmatrix} f_1 & tk \\ -tk & -\bar{\rho}\Omega^2 \end{bmatrix} = [A_{mn}], \quad \mathbf{K}_{2j}^* = \begin{bmatrix} -\bar{\rho}\Omega^2 - f_5k & -f_6k \\ f_6k & f_9 \end{bmatrix} = [B_{mn}] \tag{52}$$

The solutions of eqns (48) and (49) are, respectively, obtained as

$$\mathbf{R}_j^*(\zeta) = e^{\mathbf{K}_j^*\zeta}\mathbf{R}_j^*(0) \tag{53}$$

and

$$\mathbf{R}^{**}_j(\zeta) = e^{\mathbf{K}^{**}_j\zeta}\mathbf{R}^{**}_j(0) \tag{54}$$

Defining $\mathbf{T}_j^*(\zeta) = e^{\mathbf{K}_j^*\zeta}$ and $\mathbf{T}^{**}_j(\zeta) = e^{\mathbf{K}^{**}_j\zeta}$, the relations are obtained by employing Cayley–Hamilton theorem (Deif, 1982),

$$\mathbf{T}_j^*(\zeta) = \begin{bmatrix} \alpha_0^*(\zeta)\mathbf{I} + \alpha_2^*(\zeta)\mathbf{K}_{1j}^*\mathbf{K}_{2j}^* & \alpha_1^*(\zeta)\mathbf{K}_{1j}^* + \alpha_3^*(\zeta)\mathbf{K}_{1j}^*\mathbf{K}_{2j}^*\mathbf{K}_{1j}^* \\ \alpha_1^*(\zeta)\mathbf{K}_{2j}^* + \alpha_3^*(\zeta)\mathbf{K}_{2j}^*\mathbf{K}_{1j}^*\mathbf{K}_{2j}^* & \alpha_0^*(\zeta)\mathbf{I} + \alpha_2^*(\zeta)\mathbf{K}_{2j}^*\mathbf{K}_{1j}^* \end{bmatrix} \tag{55}$$

and

$$\mathbf{T}^{**}_j(\zeta) = \begin{bmatrix} \alpha^{**}_0(\zeta) & -\alpha^{**}_1(\zeta) f_4 k \\ \alpha^{**}_1(\zeta) f_8 & \alpha^{**}_0(\zeta) \end{bmatrix} \tag{56}$$

The parameters $\alpha^*_i(\zeta)$ in eqn (55) are determined from

$$\begin{Bmatrix} \alpha^*_0 \\ \alpha^*_1 \\ \alpha^*_2 \\ \alpha^*_3 \end{Bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 1 & -\lambda_1 & \lambda_1^2 & -\lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ 1 & -\lambda_2 & \lambda_2^2 & -\lambda_2^3 \end{bmatrix}^{-1} \begin{Bmatrix} e^{\lambda_1 \zeta} \\ e^{-\lambda_1 \zeta} \\ e^{\lambda_2 \zeta} \\ e^{-\lambda_2 \zeta} \end{Bmatrix} \tag{57}$$

where

$$\lambda_1 = \frac{\sqrt{B_0 + 2\sqrt{C_0}} + \sqrt{B_0 - 2\sqrt{C_0}}}{2}, \quad \lambda_2 = \frac{\sqrt{B_0 + 2\sqrt{C_0}} - \sqrt{B_0 - 2\sqrt{C_0}}}{2} \tag{58}$$

and

$$B_0 = A_{11}B_{11} + A_{22}B_{22} + 2A_{12}B_{12}, \quad C_0 = (A_{11}A_{22} - A_{12}^2)(B_{11}B_{22} - B_{12}^2) \tag{59}$$

The parameters $\alpha^{**}_i(\zeta)$ in eqn (56) are given by

$$\alpha^{**}_0 = \mathrm{ch}(\sqrt{f_4 f_8} k\zeta), \quad \alpha^{**}_1(\zeta) = \mathrm{sh}(\sqrt{f_4 f_8} k\zeta)/\sqrt{f_4 f_8} k \tag{60}$$

At each interface between layers, enforcing conditions of displacement and stress and utilizing eqn (53) gives

$$\mathbf{R}^*_p(d_p) = \mathbf{T}^* \mathbf{R}^*_1(0) \tag{61}$$

where

$$\mathbf{T}^* = [T^*_{mn}] = \prod_{j=p}^{1} \mathbf{T}_j(d_j) \tag{62}$$

For free vibration, the boundary conditions at the bottom and top of the laminate can be written as

$$\sigma(0) = \tau(0) = \sigma(d_p) = \tau(d_p) = 0 \tag{63}$$

Substitution of eqn (63) into (61) yields

$$\begin{bmatrix} T^*_{21} & T^*_{24} \\ T^*_{31} & T^*_{34} \end{bmatrix} \begin{Bmatrix} U(0) \\ W(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{64}$$

Setting the determinant of coefficients matrix of homogeneous eqn (64) to zero for non-trivial solutions gives the frequency equation of free axisymmetric vibration for transversely isotropic laminate. The method of evaluating frequencies and relevant mode shapes are similar to those stated in Section 3.

Similarly, enforcing the continuity conditions of potential and electric displacement at each interface between layers and using eqn (53) gives

$$\mathbf{R}^{**}_{p}(d_p) = \mathbf{T}^{**}\mathbf{R}^{**}_{1}(0) \tag{65}$$

where

$$\mathbf{T}^{**} = [T^{**}_{mn}] = \prod_{j=p}^{1} \mathbf{T}^{**}_{j}(d_j) \tag{66}$$

For free vibration, the boundary conditions at the bottom and top of the laminate can be written as

$$D_z(d_p) = D_z(0) = 0 \quad \text{or } \phi(d_p) = \phi(0) = 0 \quad \text{or } D_z(d_p) = 0, \quad \phi(0) = 0 \tag{67}$$

Substituting eqn (67) into eqn (65), respectively, yields

$$T^{**}_{12}\phi(0) = 0 \quad \text{or } T^{**}_{21}D(0) = 0 \quad \text{or } T^{**}_{11}D(0) = 0 \tag{68}$$

The frequency equation of uncoupled electric fields can be derived from eqn (68).

Equations (53) and (54) can be written as

$$\tilde{\mathbf{R}}_j(\zeta) = \tilde{\mathbf{T}}_j(\zeta)\tilde{\mathbf{R}}_j(0) \tag{69}$$

where

$$\tilde{\mathbf{R}}_j(\zeta) = \begin{Bmatrix} \mathbf{R}^{*}_{j}(\zeta) \\ \mathbf{R}^{**}_{j}(\zeta) \end{Bmatrix}, \quad \tilde{\mathbf{T}}_j(\zeta) = \begin{bmatrix} \mathbf{T}^{*}_{j}(\zeta) & 0 \\ 0 & \mathbf{T}^{**}_{j}(\zeta) \end{bmatrix} \tag{70}$$

By employing transform matrix $\mathbf{P}$, eqn (69) can also be rendered in terms of

$$\mathbf{R}_j(\zeta) = \mathbf{H}_j(\zeta)\mathbf{R}_j(0) \tag{71}$$

where $\mathbf{R}_j(\zeta)$ is defined as eqn (21) and

$$\mathbf{R}_j(\zeta) = \mathbf{P}\tilde{\mathbf{R}}_j(\zeta), \quad \mathbf{H}_j(\zeta) = \mathbf{P}\tilde{\mathbf{T}}_j(\zeta)\mathbf{P}^T \tag{72}$$

in which

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{73}$$

Based on eqn (71), the investigation on free axisymmetric vibration of hybrid circular laminate composed of piezoelectric layer and non-piezoelectric layer can be made.

5. Numerical examples

Utilizing the above mentioned analytical formulations and FEM, three numerical examples are presented. Based on the theory proposed by Sung et al. (1992), a FEM program is programmed in Fortran90 and compiled by Microsoft Fortran Powerstation 4.0. In the program, the isoparametric elements of eight-nodes are used and the eigenvalues are evaluated by sub-space iterative method.

Example 1: Consider a piezoelectric circular plate with material constants:

$c_{11} = 13.9\times10^{10}$ Pa, $c_{12} = 7.78\times10^{10}$ Pa, $c_{13} = 7.43\times10^{10}$ Pa,
$c_{33} = 11.5\times10^{10}$ Pa, $c_{44} = 2.56\times10^{10}$ Pa, $e_{15} = 12.7$ C/m^2, $e_{31} = -5.2$ C/m^2,
$e_{33} = 15.1$ C/m^2, $\varepsilon_{11} = 6.46\times10^{-9}$ F/m, $\varepsilon_{33} = 5.62\times10^{-9}$ F/m.

The non-dimensional frequencies under different boundary conditions are shown in Tables 1–4. These tables list the first three values of k and the first three frequencies for each k. Data in parentheses are obtained by FEM. Some corresponding mode shapes are shown in Figs 2–7. The results indicate that the present solutions include thickness modes and radial ones.

Example 2: Consider a single lamina transversely isotropic circular plate with elastic constants

$c_{11} = 13.9\times10^{10}$ Pa, $c_{12} = 7.78\times10^{10}$ Pa, $c_{13} = 7.43\times10^{10}$ Pa,
$c_{33} = 11.5\times10^{10}$ Pa, $c_{44} = 2.56\times10^{10}$ Pa.

Table 5 gives the lowest non-dimensional frequencies computed by the present exact method and

Table 1
The non-dimensional frequencies of a piezoelectric circular plate with rigid slipping support (Case 1)

	$k_1 = 3.83171$			$k_2 = 7.01559$			$k_3 = 10.17350$		
$t = h/a$	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.1	0.0385 (0.0386)	0.3554	1.7722	0.1231 (0.1232)	0.6454	1.8553	0.2425 (0.2430)	0.9237	1.9720
0.2	0.1451 (0.1451)	0.7034 (0.7034)	1.8768	0.4209 (0.4212)	1.2431	2.1445	0.7573	1.6846	2.4615
0.3	0.3003 (0.3004)	1.0361 (1.0361)	2.0281	0.7966 (0.7972)	1.7250	2.4976	1.3466	2.1243	2.9783
0.4	0.4865 (0.4865)	1.3434 (1.3433)	2.2072	1.2108 (1.2029)	2.0398	2.8567	1.9489	2.4589	3.4470
0.5	0.6913 (0.6913)	1.6115 (1.6117)	2.4003	1.6164 (1.6163)	2.2717	3.1951	2.5451	2.8448	3.8770
0.6	0.9070 (0.9069)	1.8278 (1.8729)	2.5980	2.0315 (2.0317)	2.5082	3.5076	3.1284	3.2881	4.3078
0.7	1.1291 (1.1289)	1.9938 (1.9940)	2.7942	2.4429 (2.4437)	2.7736	3.8037	3.6967	3.7711	4.7659
0.8	1.3544 (1.3542)	2.1286 (2.1289)	2.9847	2.8484 (2.8478)	3.0678	4.0973	4.2512	4.2785	5.2591

Table 2
The non-dimensional frequencies of a piezoelectric circular plate with rigid slipping support (Case 2)

	$k_1 = 3.83171$			$k_2 = 7.01559$			$k_3 = 10.17350$		
$t = h/a$	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.1	0.0384	0.3092	1.3880	0.1217	0.5626	1.4742	0.2379	0.8074	1.5943
0.2	0.1432	0.6135	1.4964	0.4102	1.0920	1.7718	0.7345	1.5023	2.1022
0.3	0.2938	0.9071	1.6520	0.7724	1.5422	2.1404	1.3063	1.9765	2.6646
0.4	0.4733	1.1827	1.8366	1.1654	1.8795	2.5289	1.8943	2.3453	3.2055
0.5	0.6707	1.4317	2.0377	1.5695	2.1433	2.9117	2.4763	2.7402	3.7078
0.6	0.8793	1.6468	2.2474	1.9750	2.3969	3.2767	3.0437	3.1809	4.1928
0.7	1.0947	1.8273	2.4601	2.3766	2.6685	3.6231	3.5947	3.6560	4.6877
0.8	1.3140	1.9816	2.6719	2.7715	2.9628	3.9587	4.1314	4.1524	5.2058

Table 3
The non-dimensional frequencies of a piezoelectric circular plate with elastic simple support (Case 1)

	$k_1 = 2.40483$			$k_2 = 5.52008$			$k_4 = 8.65373$		
$t = h/a$	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.1	0.0154 (0.0154)	0.2235	1.7494	0.0782 (0.0782)	0.5101	1.8112	0.1814 (0.1816)	0.7912	1.9123
0.2	0.0600 (0.0600)	0.4452 (0.4452)	1.7933	0.2800 (0.2801)	0.9977	2.0084	0.5906	1.4879	2.3060
0.3	0.1297 (0.1297)	0.6633 (0.6633)	1.8618	0.5508 (0.5509)	1.4347	2.2684	1.0789	1.9601	2.7505
0.4	0.2195 (0.2195)	0.8757 (0.8757)	1.9496	0.8551 (0.8554)	1.7814	2.5511	1.5890	2.2568	3.1736
0.5	0.3246 (0.3245)	1.0803 (1.0803)	2.0516	1.1747 (1.1744)	2.0230	2.8335	2.1008	2.5505	3.5580
0.6	0.4409 (0.4408)	1.2743 (1.2743)	2.1636	1.5004 (1.5001)	2.2088	3.1034	2.6063	2.8884	3.9210
0.7	0.5653 (0.5652)	1.4544 (1.4543)	2.2821	1.8275 (1.8273)	2.3887	3.3567	3.1020	3.2668	4.2876
0.8	0.6957 (0.6955)	1.6167 (1.6166)	2.4044	2.1533 (2.1533)	2.5832	3.5959	3.5870	3.6746	4.6738

Table 4
The non-dimensional frequencies of a piezoelectric circular plate with elastic simple support (Case 2)

	$k_1 = 2.40483$			$k_2 = 5.52008$			$k_3 = 8.65373$		
$t = h/a$	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.1	0.0154	0.1944	1.3641	0.0776	0.4442	1.4285	0.1785	0.6907	1.5329
0.2	0.0596	0.3876	1.4099	0.2741	0.8730	1.6318	0.5736	1.3153	1.9391
0.3	0.1281	0.5783	1.4808	0.5353	1.2662	1.9000	1.0459	1.7895	2.4123
0.4	0.2156	0.7651	1.5713	0.8290	1.5990	2.1972	1.5427	2.1267	2.8870
0.5	0.3172	0.9464	1.6761	1.1390	1.8604	2.5033	2.0426	2.4408	3.3359
0.6	0.4293	1.1202	1.7915	1.4563	2.0729	2.8064	2.5358	2.7839	3.7584
0.7	0.5493	1.2843	1.9142	1.7757	2.2708	3.0996	3.0181	3.1599	4.1706
0.8	0.6750	1.4366	2.0421	2.0939	2.4746	3.3804	3.4884	3.5613	4.5896

FEM. A good agreement is reached. This example and the next employ the characteristic eqn (64) to evaluate the frequencies.

Example 3: A three-ply laminated circular plate. The first and third laminae are made of isotropic material of Young's modulus $E = 2.1 \times 10^{11}$ Pa and Poisson's ratio $\mu = 0.3$ and the second one is made of transversely isotropic material. Its elastic constants are the same as those of Example 2. Three laminae have thicknesses of $h_1 = h_3 = h/4$ and $h_2 = h/2$ and their densities are $\rho_1 = \rho_3 = 7.8 \times 10^3$ kg/m^3 and $\rho_2 = 7.5 \times 10^3$ kg/m^3, respectively. Table 6 gives the lowest non-dimensional frequencies computed by the present exact method and FEM. A good agreement is also reached. Tables 7 and 8 list the first three values of k and the first three frequencies for each k.

6. Conclusions

(1) The state-space-based method and the finite Hankel transform are employed to analyze the free axisymmetric vibration of piezoelectric laminated circular plates. It is found that exact solutions can be obtained for two kinds of boundary conditions. These solutions can be used to examine the validity of various plate theories and numerical calculation software.
(2) The numerical examples show that the non-dimensional frequencies increase with the increase of thickness-to-span ratio and frequencies for elastic simple support boundary condition are obviously smaller than those for rigid slipping supported one.
(3) The comparison between the present results and FEM results shows that if all frequencies for distinct k are arranged in order of their values, good agreement can be observed.

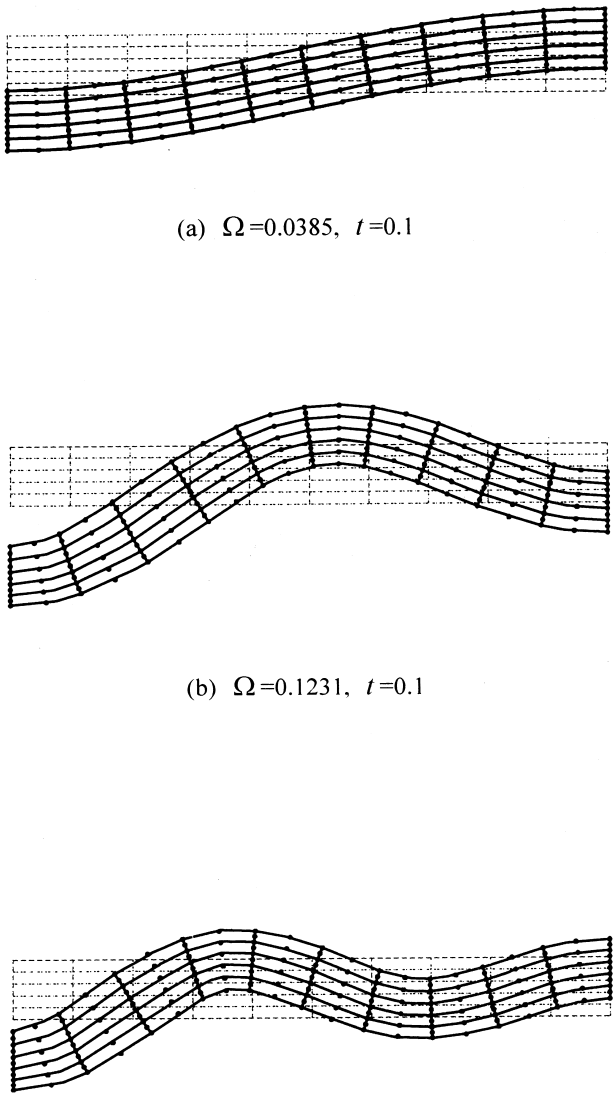
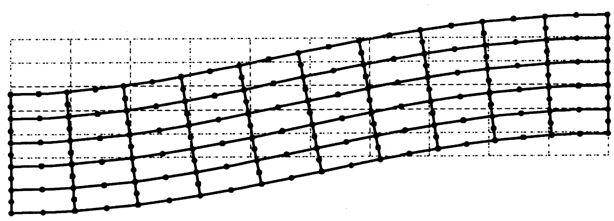
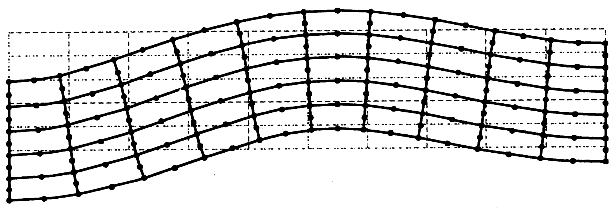


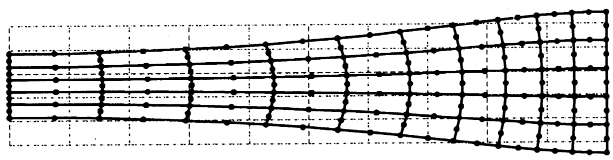
Fig. 2. The mode shapes of a piezoelectric circular plate with rigid slipping support (Case 1) (---, initial mesh; —, the present theory; ●, FEM).



(a) $\Omega=0.1451$, $t=0.2$

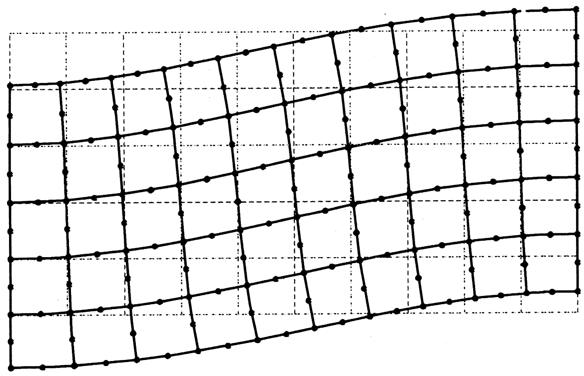


(b) $\Omega=0.4209$, $t=0.2$

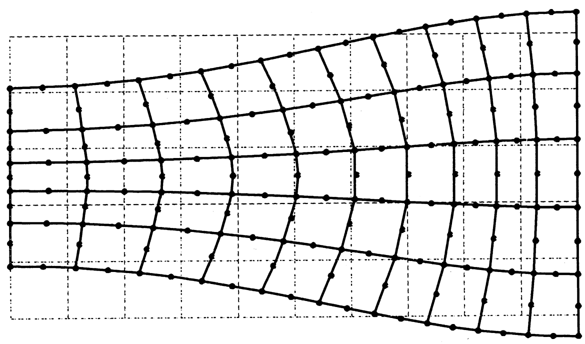


(c) $\Omega=0.7034$, $t=0.2$

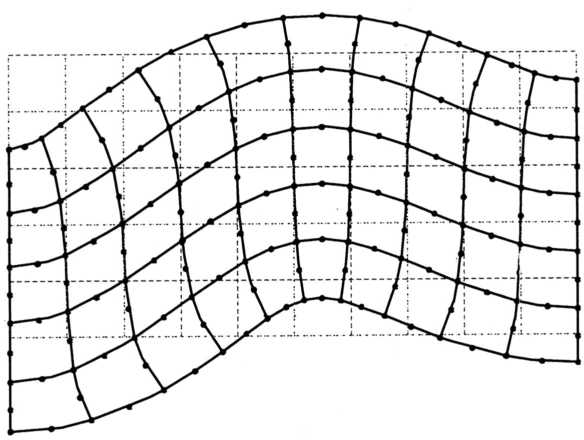
Fig. 3. The mode shapes of a piezoelectric circular plate with rigid slipping support (Case 1) (---, initial mesh; —, the present theory; •, FEM).



(a) $\Omega=0.6913$, $t=0.5$

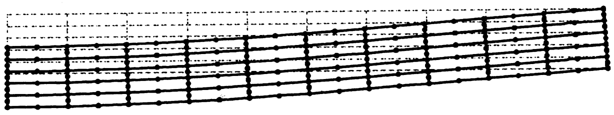


(b) $\Omega=1.6115$, $t=0.5$

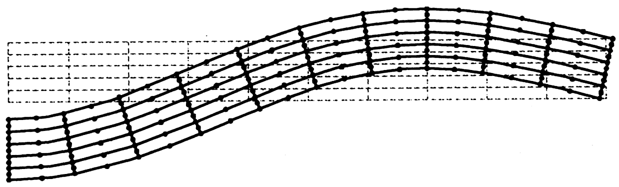


(c) $\Omega=1.6164$, $t=0.5$

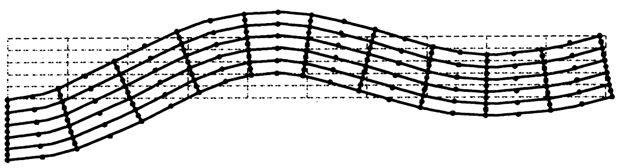
Fig. 4. The mode shapes of a piezoelectric circular plate with rigid slipping support (Case 1) (- - -, initial mesh; —, the present theory; •, FEM).



(a) Ω=0.0154, t=0.1

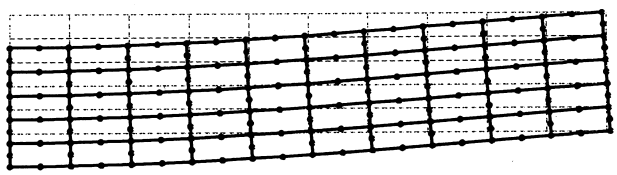


(b) Ω=0.0782, t=0.1

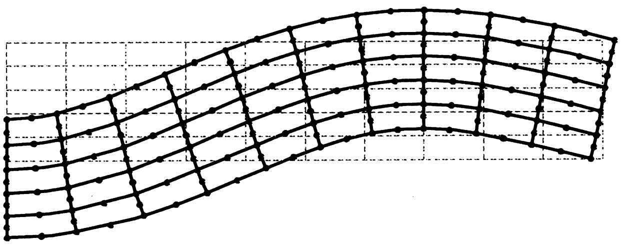


(c) Ω=0.1814, t=0.1

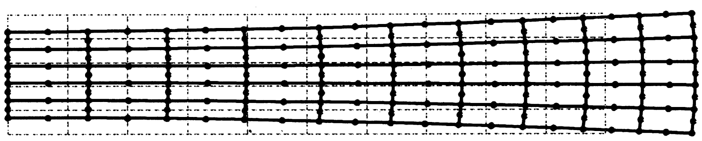
Fig. 5. The mode shapes of a piezoelectric circular plate with elastic simple support (Case 1) (---, initial mesh; —, the present theory; •, FEM).



(a) Ω=0.0600, t=0.2

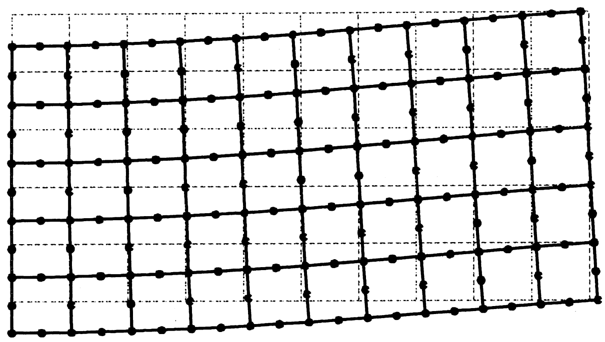


(b) Ω=0.2800, t=0.2

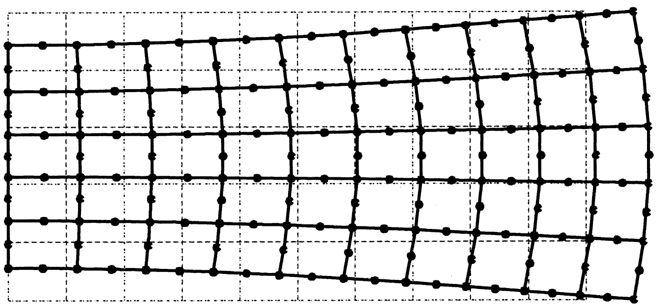


(c) Ω=0.4452, t=0.2

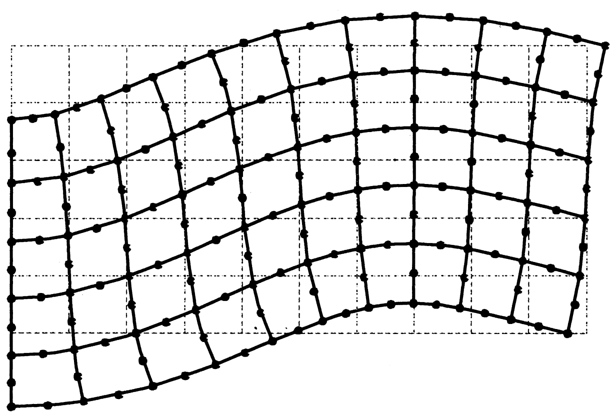
Fig. 6. The mode shapes of a piezoelectric circular plate with elastic simple support (Case 1) (- - -, initial mesh; —, the present theory; •, FEM).



(a) $\Omega = 0.3246$, $t = 0.5$



(b) $\Omega = 1.0803$, $t = 0.5$



(c) $\Omega = 1.1747$, $t = 0.5$

Fig. 7. The mode shapes of a piezoelectric circular plate with elastic simple support (Case 1) (- - -, initial mesh; —, the present theory; •, FEM).

Table 5
The first non-dimensional frequency of a single lamina circular plate

	Rigid slipping support		Elastic simple support	
$t = h/a$	Present	FEM	Present	FEM
0.1	0.0332	0.0333	0.0133	0.0133
0.2	0.1233	0.1233	0.0516	0.0516
0.3	0.2505	0.2505	0.1104	0.1105
0.4	0.3985	0.3986	0.1847	0.1848
0.5	0.5573	0.5574	0.2699	0.2700
0.6	0.7214	0.7216	0.3625	0.3626
0.7	0.8879	0.8881	0.4600	0.4601
0.8	1.0550	1.0558	0.5607	0.5608

Table 6
The first non-dimensional frequency of a three-ply circular plate

	Rigid slipping support		Elastic simple support	
$t = h/a$	Present	FEM	Present	FEM
0.1	0.0353	0.0353	0.0143	0.0143
0.2	0.1248	0.1249	0.0541	0.0542
0.3	0.2419	0.2420	0.1125	0.1125
0.4	0.3706	0.3707	0.1824	0.1825
0.5	0.5044	0.5045	0.2592	0.2593
0.6	0.6410	0.6411	0.3399	0.3399
0.7	0.7798	0.7799	0.4228	0.4229
0.8	0.9209	0.9211	0.5073	0.5073

Table 7
The non-dimensional frequencies of a three-ply circular plate with rigid slipping support

	$k_1 = 3.83171$			$k_2 = 7.01559$			$k_3 = 10.17350$		
$t = h/a$	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.1	0.0353	0.2906	1.0642	0.1071	0.5262	1.1757	0.1997	0.7477	1.3261
0.2	0.1248	0.5728	1.2039	0.3263	0.9883	1.5411	0.5465	1.2607	1.9230
0.3	0.2419	0.8348	1.3969	0.5714	1.2808	1.9656	0.9160	1.4598	2.4959
0.4	0.3706	1.0580	1.6179	0.8252	1.4200	2.3704	1.3029	1.6630	2.8826
0.5	0.5044	1.2225	1.8505	1.0870	1.5404	2.6960	1.7085	1.9562	3.1086
0.6	0.6410	1.3283	2.0825	1.3576	1.6980	2.9203	2.1296	2.3230	3.2909
0.7	0.7798	1.3998	2.3031	1.6371	1.8995	3.0753	2.5578	2.7387	3.4885
0.8	0.9209	1.4620	2.5024	1.9246	2.1383	3.2032	2.9782	3.1848	3.7270

Table 8
The non-dimensional frequencies of a three-ply circular plate with elastic simple support

	$k_1 = 2.40483$			$k_2 = 5.52008$			$k_3 = 8.65373$		
$t = h/a$	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.1	0.0143	0.1829	1.0325	0.0698	0.4166	1.1171	0.1533	0.6431	1.2499
0.2	0.0541	0.3638	1.0930	0.2272	0.8053	1.3721	0.4394	1.1511	1.7375
0.3	0.1125	0.5405	1.1842	0.4133	1.1180	1.6921	0.7484	1.3852	2.2552
0.4	0.1824	0.7101	1.2976	0.6084	1.3074	2.0280	1.0694	1.5316	2.6778
0.5	0.2592	0.8684	1.4261	0.8083	1.4125	2.3456	1.4038	1.7288	2.9498
0.6	0.3399	1.0104	1.5645	1.0313	1.5041	2.6165	1.7516	1.9913	3.1281
0.7	0.4228	1.1304	1.7087	1.2232	1.6148	2.8215	2.1102	2.3050	3.2825
0.8	0.5073	1.2253	1.8854	1.4390	1.7530	2.9710	2.4743	2.6547	3.4474

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